

# BLOW-UP CRITERION FOR THE 3D NON-RESISTIVE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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**ABSTRACT.** In this paper, we prove a blow-up criterion in terms of the magnetic field  $H$  and the mass density  $\rho$  for the strong solutions to the 3D compressible isentropic MHD equations with zero magnetic diffusion and initial vacuum. More precisely, we show that the upper bounds of  $(H, \rho)$  control the possible blow-up (see [25][30]) for strong solutions, which means that if a solution of the compressible isentropic non-resistive MHD equations is initially smooth and loses its regularity at some later time, then the formation of singularity must be caused by losing the upper bound of  $H$  or  $\rho$  as the blow-up time approaches.

## 1. INTRODUCTION

Magnetohydrodynamics is that part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled. In 3D space, the compressible isentropic MHD equations in a domain  $\Omega \subset \mathbb{R}^3$  can be written as

$$\begin{cases} H_t - \operatorname{rot}(u \times H) = -\operatorname{rot}\left(\frac{1}{\sigma}\operatorname{rot}H\right), \\ \operatorname{div}H = 0, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}\mathbb{T} + \operatorname{rot}H \times H. \end{cases} \quad (1.1)$$

In this system,  $x \in \Omega$  is the spatial coordinate;  $t \geq 0$  is the time;  $H = (H^1, H^2, H^3)$  is the magnetic field;  $0 < \sigma \leq \infty$  is the electric conductivity coefficient;  $\rho$  is the mass density;  $u = (u^1, u^2, u^3) \in \mathbb{R}^3$  is the velocity of fluids;  $P$  is the pressure satisfying

$$P = A\rho^\gamma, \quad A > 0, \quad \gamma > 1, \quad (1.2)$$

where  $A$  is a constant and  $\gamma$  is the adiabatic index;  $\mathbb{T}$  is the viscosity stress tensor:

$$\mathbb{T} = 2\mu D(u) + \lambda \operatorname{div}u \mathbb{I}_3, \quad D(u) = \frac{\nabla u + (\nabla u)^\top}{2}, \quad (1.3)$$

where  $D(u)$  is the deformation tensor,  $\mathbb{I}_3$  is the  $3 \times 3$  unit matrix,  $\mu$  is the shear viscosity coefficient,  $\lambda + \frac{2}{3}\mu$  is the bulk viscosity coefficient,  $\mu$  and  $\lambda$  are both real constants,

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0, \quad (1.4)$$

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which ensures the ellipticity of the Lamé operator (see (1.9)). Although the electric field  $E$  doesn't appear in system (1.1), it is indeed induced according to a relation

$$E = \frac{1}{\sigma} \text{rot} H - u \times H$$

by moving the conductive flow in the magnetic field.

The MHD system (1.1) describes the macroscopic behavior of electrically conducting compressible (isentropic) fluids in a magnetic field. It is reasonable to assume that there is no magnetic diffusion (i.e.  $\sigma = +\infty$ ) when the conducting fluid considered is of a very high conductivity, which occurs frequently in many cosmical and geophysical problems. Then we need to consider the following system:

$$\begin{cases} H_t - \text{rot}(u \times H) = 0, \\ \text{div} H = 0, \\ \rho_t + \text{div}(\rho u) = 0, \\ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \text{div} \mathbb{T} + \text{rot} H \times H, \end{cases} \quad (1.5)$$

which is the so called viscous and non-resistive MHD equations (see [5][11][13][19][20]).

The aim of this paper is to give a blow-up criterion of strong solutions to the initial boundary value problem (IBVP): system (1.5) in a bounded, smooth domain  $\Omega \subset \mathbb{R}^3$  with the initial boundary value conditions:

$$(H, \rho, u)|_{t=0} = (H_0(x), \rho_0(x), u_0(x)), \quad x \in \Omega; \quad u|_{\partial\Omega} = 0. \quad (1.6)$$

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$$\begin{aligned} D^{k,r} &= \{f \in L^1_{loc}(\Omega) : |f|_{D^{k,r}} = |\nabla^k f|_{L^r} < +\infty\}, \quad D^k = D^{k,2}, \\ D^1_0 &= \{f \in L^6(\Omega) : |f|_{D^1} = |\nabla f|_{L^2} < \infty \text{ and } f|_{\partial\Omega} = 0\}, \quad \|(f, g)\|_X = \|f\|_X + \|g\|_X, \\ \|f\|_{W^{m,r}} &= \|f\|_{W^{m,r}(\Omega)}, \quad \|f\|_s = \|f\|_{H^s(\Omega)}, \quad |f|_p = \|f\|_{L^p(\Omega)}, \\ |f|_{D^{k,r}} &= \|f\|_{D^{k,r}(\Omega)}, \quad |f|_{D^k} = \|f\|_{D^k(\Omega)}, \quad \mathbb{A} : \mathbb{B} = \sum_{ij} a_{ij} b_{ij}. \end{aligned}$$

A detailed study of homogeneous Sobolev spaces can be found in [12].

As has been observed in [10], which proved the existence of unique local strong solutions with initial vacuum to IBVP (1.5)-(1.6), in order to make sure that the IBVP (1.5)-(1.6) with initial vacuum is well-posed, the lack of a positive lower bound of the initial mass density  $\rho_0$  should be compensated with some initial layer compatibility condition on the initial data  $(H_0, \rho_0, u_0, P_0)$ :

**Theorem 1.1.** [10] *Let constant  $q \in (3, 6]$ . If  $(H_0, \rho_0, u_0, P_0)$  satisfies*

$$(H_0, \rho_0, P_0) \in H^1 \cap W^{1,q}, \quad \rho_0 \geq 0, \quad u_0 \in D^1_0 \cap D^2, \quad (1.7)$$

*and the compatibility condition*

$$Lu_0 + \nabla P_0 - \text{rot} H_0 \times H_0 = \sqrt{\rho_0} g_1 \quad (1.8)$$

for some  $g_1 \in L^2$ ,  $P_0 = A\rho_0^\gamma$ , and

$$Lu_0 = -\mu\Delta u_0 - (\lambda + \mu)\nabla \operatorname{div} u_0, \quad (1.9)$$

then there exists a time  $T_*$  and a unique solution  $(H, \rho, u, P)$  to IBVP (1.5)-(1.6) satisfying

$$\begin{aligned} (H, \rho, P) &\in C([0, T_*]; H^1 \cap W^{1,q}), \quad u \in C([0, T_*]; D_0^1 \cap D^2) \cap L^2([0, T_*]; D^{2,q}), \\ u_t &\in L^2([0, T_*]; D_0^1), \quad \sqrt{\rho}u_t \in L^\infty([0, T_*]; L^2). \end{aligned}$$

Some analogous existence theorems of the unique local strong solutions to the compressible Navier-Stokes equations have been previously established by CHo-Choe-Kim in [7][8][9]. In 3D space, Huang-Li-Xin obtained the well-posedness of global classical solutions with small energy but possibly large oscillations and vacuum for Cauchy problem in [14] or IBVP in [15] to the isentropic flow. For compressible MHD equations, when  $0 < \sigma < +\infty$ , the global smooth solution near the constant state in one-dimensional space was studied in Kawashima-Okada [18]; recently, in 3D space, the similar result to [14] has been obtained in Li-Xu-Zhang [21]. However, for  $\sigma = +\infty$ , at least as far as I know, there are few results on the global existence of strong solutions with initial vacuum. The non-global existence in the whole space  $\mathbb{R}^3$  has been proved in [25] for the classical solution to isentropic MHD equations as follows:

**Theorem 1.2.** [25] *Assume that  $\gamma \geq \frac{6}{5}$ , if the momentum  $\int_{\mathbb{R}^3} \rho_0 u_0 dx \neq 0$ , then there exists no global classical solutions to (1.5)-(1.6) with conserved mass, momentum and total energy.*

Then these motivate us to consider that the local strong solutions to (1.5)-(1.6) may cease to exist globally, or what is the key point to make sure that the solution obtained in Theorem 1.1 could become a global one? If the blow-up happens, we want to know the mechanism of breakdown and the structure of singularities? The similar question has been studied for the incompressible Euler equation by Beale-Kato-Majda (BKM) in their pioneering work [3], which showed that the  $L^\infty$ -bound of vorticity  $\operatorname{rot} u$  must blow up if we assume that the life span of the corresponding strong solution is finite. Later, Ponce [24] rephrased the BKM-criterion in terms of the deformation tensor  $D(u)$ , and the same result as [24] has been proved by Huang-Li-Xin [16] for compressible isentropic Navier-Stokes equations, which can be shown: if  $0 < \bar{T} < +\infty$  is the maximum existence time for strong solution, then

$$\limsup_{T \rightarrow \bar{T}} \int_0^T |D(u)|_{L^\infty(\Omega)} dt = \infty. \quad (1.10)$$

Recently, the similar blow-up criterions as (1.10) have been obtained for the 3D compressible isentropic MHD equations in Xu-Zhang [32], which can be shown:

$$\limsup_{T \rightarrow \bar{T}} \int_0^T |\nabla u|_{L^\infty(\Omega)} dt = \infty. \quad (1.11)$$

Some similar results also can be seen in Chen-Liu [22] or Lu-Du-Yao [23].

Moreover, for the strong solutions with initial vacuum to 3D compressible isentropic Navier-stokes equations, Sun-Wang-Zhang [28] proved

$$\limsup_{T \rightarrow \bar{T}} |\rho|_{L^\infty([0, T] \times \Omega)} = \infty,$$

under the physical assumption (1.4) and  $\lambda < 7\mu$ .

In the following theorem, under the physical assumption (1.4) and  $3\lambda < 29\mu$ , we show that the  $L^\infty$  norms of the magnetic field  $H$  and the mass density  $\rho$  control the possible blow-up (see [25][30]) for strong solutions, which means that if a solution of the compressible MHD equations is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the upper bound of  $H$  or  $\rho$  as the critical time approaches. The arguments used in [16][28] can not be applied to our case directly. The first reason is that we will relax the assumption  $\lambda < 7\mu$  to  $3\lambda < 29\mu$ . The second reason is the presence of magnetic momentum flux density tensor

$$\frac{1}{2}|H|^2 I_3 - H \otimes H$$

in momentum equations (1.1)<sub>4</sub>. To deal with this nonlinear term, we need to control the norm  $|\nabla H|_2$ , which is difficult to be bounded by  $|D(u)|_{L^1(0,T;L^\infty)}$  because of the strong coupling between  $u$  and  $H$  in magnetic equations (1.1)<sub>1</sub>, and the lack of smooth mechanism of  $H$  for the case  $\sigma = +\infty$ . These are unlike those for  $(|\rho|_\infty, |\nabla \rho|_2)$ , which can be totally determined by  $|\operatorname{div} u|_{L^1(0,T;L^\infty)}$  due to the scalar hyperbolic structure of the continuity equation (1.1)<sub>1</sub> in [16][28]. So some new arguments need to be introduced to improve the results obtained above for system (1.1).

**Theorem 1.3.** *Let the viscosity coefficients  $(\mu, \lambda)$  satisfy*

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0, \quad 3\lambda < 29\mu, \quad (1.12)$$

*and  $(H_0, \rho_0, u_0, P_0)$  satisfy (1.7)-(1.8). If  $(H, \rho, u, P)$  is a strong solution to IBVP (1.5)-(1.6) obtained in Theorem 1.1, and  $0 < \bar{T} < \infty$  is the maximal time of its existence, then*

$$\limsup_{T \rightarrow \bar{T}} (|\rho|_{L^\infty([0,T] \times \Omega)} + |H|_{L^\infty([0,T] \times \Omega)}) = \infty. \quad (1.13)$$

**Remark 1.1.** *We introduce the main ideas of our proof for Theorem 1.3, some of which are inspired by the arguments used in [16][28][29].*

*I) We improve the methods used in [28][29] to obtain the estimate (3.5) under the assumption (1.12). In order to prove (3.5), the restriction  $\lambda < 7\mu$  plays an key role in the analysis shown in [28], and actually, it is only used to get the upper bound of  $\int_\Omega \rho |u(t)|^r dx$  for some  $r > 3$ . However, Wen-Zhu [29] obtain the upper bound of  $\int_\Omega \rho |u(t)|^r dx$  under the assumption  $3\lambda < 29\mu$ , which as a byproduct extends the conclusions obtained in [28]. Compared with [29], we need to deal with the magnetic term appearing in the momentum equations, and due to the initial vacuum, we obtain the upper bound of  $\int_\Omega \rho |u(t)|^r dx$  for  $r \in (3, 7/2)$  under the assumption (1.12).*

*In order to get a restriction of  $\mu$  and  $\lambda$  as better as possible, the crucial ingredient to relax the additional restrictions to  $3\lambda < 29\mu$  has been observed (see [29]) that*

$$|\nabla u|^2 = |u|^2 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u||^2, \quad (1.14)$$

*for  $|u| > 0$ , and thus*

$$\int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla u|^2 dx \geq (1 + \phi(\epsilon_0, \epsilon_1, r)) \int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla |u||^2 dx, \quad (1.15)$$

if

$$\int_{\Omega \cap |u| > 0} |u|^r \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 dx \geq \phi(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla |u||^2 dx \quad (1.16)$$

for some positive function  $\phi(\epsilon_0, \epsilon_1, r)$  near  $r = 3$ . The details can be seen in Lemma 3.2.

II) If  $|\rho|_{L^\infty([0,T];L^\infty(\Omega))}$  and  $|H|_{L^\infty([0,T];L^\infty(\Omega))}$  are bounded, we can obtain a high integrability of velocity  $u$ , which can be used to control the nonlinear terms (See Lemmas 3.2-3.3). The argument used in [28] is introduced to control the upper bound of  $|\nabla u|_2$ , and a important observation has been shown in Lemma 3.3 that

$$\begin{cases} \mathbb{B} = H_t \otimes H + H \otimes H_t \\ = (H^i H^k \partial_k u^j + H^j H^k \partial_k u^i - H^i H^j \partial_k u^k)_{(ij)} - \operatorname{div}((H \otimes H) \otimes u), \\ \mathbb{C} = H \cdot H_t = H \cdot (H \cdot \nabla u - u \cdot \nabla H - H \operatorname{div} u) \\ = (H \cdot \nabla u \cdot H - \frac{1}{2} |H|^2 \operatorname{div} u) - \frac{1}{2} \operatorname{div}(u |H|^2), \end{cases} \quad (1.17)$$

from which, we successfully avoid the difficulty coming from the strong coupling between the magnetic field and velocity when the magnetic diffusion vanishes.

The next difficulty is to control the mass density  $\rho$  and the magnetic field  $H$ , which both satisfy hyperbolic equations. To do this, we need to make sure that the velocity  $u$  is bounded in  $L^1([0,T];D^{1,\infty}(\Omega))$ . On the other hand, in order to prove  $u \in L^1([0,T];D^{1,\infty}(\Omega))$ , we have to obtain some priori bounds for  $\nabla \rho$  and  $\nabla H$ . Furthermore, the magnetic term in the momentum equation will bring extra difficulty to us. However, via using the argument from [17] and the structure of the magnetic equations, in Lemma 3.5, we show that

$$\begin{aligned} \Lambda &= \int_{\Omega} \dot{u} \cdot \left[ \operatorname{div}(H \otimes H - \frac{1}{2} |H|^2 I_3)_t + \operatorname{div}(\operatorname{div}(H \otimes H - \frac{1}{2} |H|^2 I_3) \otimes u) \right] dx \\ &= \int_{\Omega} \partial_k u^k H^i H^j \partial_j \dot{u}^i dx + \int_{\Omega} \left( -\frac{1}{2} \partial_j u^j |H^k|^2 \partial_i \dot{u}^i \right) dx. \end{aligned} \quad (1.18)$$

Then we get the cancelation to the derivatives  $(\nabla \rho, \nabla H)$  during our computation, which brings us the desired result.

The rest of this paper is organized as follows. In Section 2, we give some important lemmas which will be used frequently in our proof. In Section 3, we give the proof for the blow-up criterion (1.13).

## 2. PRELIMINARY

In this section, we give some important lemmas which will be used frequently in our proof. The first one is some Sobolev inequalities:

**Lemma 2.1.** *For  $l \in (3, \infty)$ , there exists some generic constant  $C > 0$  that may depend on  $l$  such that for  $f \in D_0^1(\Omega)$ ,  $g \in D_0^1 \cap D^2(\Omega)$  and  $h \in W^{1,l}(\Omega)$ , we have*

$$|f|_6 \leq C|f|_{D_0^1}, \quad |g|_\infty \leq C|g|_{D_0^1 \cap D^2}, \quad |h|_\infty \leq C\|h\|_{W^{1,l}}. \quad (2.1)$$

Next we consider the following boundary value problem for the Lamé operator  $L$ :

$$\begin{cases} -\mu\Delta U - (\mu + \lambda)\nabla\operatorname{div}U = F, & \text{in } \Omega, \\ U(t, x) = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.2)$$

where  $U = (U^1, U^2, U^3)$ ,  $F = (F^1, F^2, F^3)$ . It is well known that under the assumption (1.4), (2.2)<sub>1</sub> is a strongly elliptic system. If  $F \in W^{-1,2}(\Omega)$ , then there exists a unique weak solution  $U \in D_0^1(\Omega)$ . We begin with recalling various estimates for this system in  $L^l(\Omega)$  spaces, which can be seen in [1].

**Lemma 2.2.** *Let  $l \in (1, +\infty)$  and  $u$  be a solution of (2.2). There exists a constant  $C$  depending only on  $\lambda$ ,  $\mu$ ,  $l$  and  $\Omega$  such that the following estimates hold:*

(1) *if  $F \in L^l(\Omega)$ , then we have*

$$\|U\|_{W^{2,l}} \leq C\|F\|_l; \quad (2.3)$$

(2) *if  $F \in W^{-1,l}(\Omega)$  (i.e.,  $F = \operatorname{div}f$  with  $f = (f_{ij})_{3 \times 3}$ ,  $f_{ij} \in L^l(\Omega)$ ), then we have*

$$\|U\|_{W^{1,l}} \leq C\|f\|_l; \quad (2.4)$$

(3) *if  $F = \operatorname{div}f$  with  $f_{ij} = \partial_k h_{ij}^k$  and  $h_{ij}^k \in W_0^{1,l}(\Omega)$  for  $i, j, k = 1, 2, 3$ , then we have*

$$|U|_l \leq C|h|_l. \quad (2.5)$$

Moreover, we need an endpoint estimate for  $L$  in the case  $l = \infty$ . Let  $BMO(\Omega)$  stands for the John-Nirenberg space of bounded mean oscillation whose norm is defined by:

$$\|F\|_{BMO(\Omega)} = \|f\|_{L^2(\Omega)} + [f]_{[BMO]}, \quad (2.6)$$

with

$$\begin{cases} [f]_{[BMO]} = \sup_{x \in \Omega, r \in (0, d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}| dy, \\ f_{\Omega_r(x)} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) dy, \end{cases} \quad (2.7)$$

where  $\Omega_r(x) = B_r(x) \cap \Omega$ ,  $B_r(x)$  is the ball with center  $x$  and radius  $r$ , and  $d$  is the diameter of  $\Omega$ .  $|\Omega_r(x)|$  denotes the Lebesgue measure of  $\Omega_r(x)$ . Note that

$$[f]_{[BMO]} \leq 2\|f\|_\infty. \quad (2.8)$$

**Lemma 2.3.** *If  $F = \operatorname{div}f$  with  $f = (f_{ij})_{3 \times 3}$ ,  $f_{ij} \in L^\infty(\Omega) \cap L^2(\Omega)$ , then  $\nabla U \in BMO(\Omega)$  and there exists a constant  $C$  depending only on  $\lambda$ ,  $\mu$  and  $\Omega$  such that*

$$|\nabla U|_{[BMO]} \leq C(\|f\|_\infty + \|f\|_2). \quad (2.9)$$

Because  $\Omega$  is a bounded domain with smooth boundary, the estimate (2.9) can be found in [1] for a more general setting. In the next lemma, we will give a variant of the Brezis-Wagner inequality [6], which also can be seen in [28].

**Lemma 2.4.** [6] *Let  $\Omega$  be a bounded Lipschitz domain and  $f \in W^{1,l}(\Omega)$  with  $l \in (3, \infty)$ . There exists a constant  $C$  depending on  $l$  and the Lipschitz property of  $\Omega$  such that*

$$\|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{BMO(\Omega)} \ln(e + \|\nabla f\|_l)). \quad (2.10)$$

Finally, for  $(H, u) \in C^1(\Omega)$ , there are some formulas based on  $\operatorname{div}H = 0$ :

**Lemma 2.5.** *Let  $(H, \rho, u, P)$  be the unique strong solution obtained in Theorem 1.1 to IBVP (1.5)–(1.6) in  $[0, T) \times \Omega$ , then we have*

$$\begin{cases} \operatorname{rot}(u \times H) = (H \cdot \nabla)u - (u \cdot \nabla)H - H \operatorname{div} u, \\ \operatorname{rot} H \times H = \operatorname{div} \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right) = -\frac{1}{2} \nabla |H|^2 + H \cdot \nabla H. \end{cases} \quad (2.11)$$

*Proof.* It follows immediately from the following equality:

$$\begin{cases} a \times \operatorname{rot} a = \frac{1}{2} \nabla (|a|^2) - a \cdot \nabla a, \\ \operatorname{rot}(a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + (\operatorname{div} b)a - (\operatorname{div} a)b, \end{cases}$$

based on the fact that  $\operatorname{div} H = 0$ . □

### 3. BLOW-UP CRITERION (1.13) FOR STRONG SOLUTIONS

Now we prove (1.13). Let  $(H, \rho, u, P)$  be the unique strong solution obtained in Theorem 1.1 to IBVP (1.5)–(1.6) in  $[0, \bar{T}) \times \Omega$ . Due to  $P = A\rho^\gamma$ , we show that  $P$  satisfies

$$P_t + u \cdot \nabla P + \gamma P \operatorname{div} u = 0, \quad P_0 \in H^2 \cap W^{2,q}. \quad (3.1)$$

We first give the standard energy estimate that

**Lemma 3.1.**

$$|\sqrt{\rho}u(t)|_2^2 + |H(t)|_2^2 + |P(t)|_1 + \int_0^T |\nabla u(t)|_2^2 dt \leq C, \quad \text{for } 0 \leq t \leq T,$$

where  $C$  only depends on  $C_0, \mu, \lambda, A, \gamma, \Omega$  and  $T$  (any  $T \in (0, \bar{T})$ ).

*Proof.* We first show that

$$\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \rho |u|^2 + \frac{P}{\gamma-1} + \frac{1}{2} H^2 \right) dx + \int_\Omega (\mu |\nabla u|^2 + (\lambda + \mu) (\operatorname{div} u)^2) dx = 0. \quad (3.2)$$

Actually, (3.2) is classical, which can be shown by multiplying (1.5)<sub>4</sub> by  $u$ , (1.5)<sub>3</sub> by  $\frac{|u|^2}{2}$  and (1.5)<sub>1</sub> by  $H$ , then summing them together and integrating the resulting equation over  $\Omega$  by parts, where we have used the fact

$$\int_\Omega \operatorname{rot} H \times H \cdot u dx = \int_\Omega -\operatorname{rot}(u \times H) \cdot H dx. \quad (3.3)$$

□

Next we assume that the opposite of (1.13) holds, i.e.,

$$\limsup_{T \rightarrow \bar{T}} \left( |\rho|_{L^\infty([0, T] \times \Omega)} + |H|_{L^\infty([0, T] \times \Omega)} \right) = C_0 < \infty. \quad (3.4)$$

Now based on (3.4), we can improve the energy estimate obtained in Lemma 3.1.

**Lemma 3.2.** *If (1.12) holds, then there exists  $r \in (3, \frac{7}{2})$  such that*

$$\int_\Omega \rho |u(t)|^r dx \leq C, \quad \text{for } 0 \leq t \leq T, \quad (3.5)$$

where  $C$  only depends on  $C_0, \mu, \lambda, A, \gamma, \Omega$  and  $T$  (any  $T \in (0, \bar{T})$ ).

*Proof.* For any  $\lambda$  satisfying (1.12), there must exists a sufficiently small constant  $\alpha_\lambda > 0$ :

$$3\lambda < (29 - \alpha_\lambda)\mu. \quad (3.6)$$

So we only need to show that (3.5) holds under the assumption (3.6).

First, multiplying (1.5)<sub>4</sub> by  $r|u|^{r-2}u$  ( $r \geq 3$ ) and integrating over  $\Omega$ , we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho |u|^r dx + \int_{\Omega} H_r dx \\ &= -r(r-2)(\mu + \lambda) \int_{\Omega} \operatorname{div} u |u|^{r-3} u \cdot \nabla |u| dx \\ & \quad + \int_{\Omega} r P \operatorname{div} (|u|^{r-2} u) dx - \int_{\Omega} r (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla (|u|^{r-2} u) dx, \end{aligned} \quad (3.7)$$

where

$$H_r = r|u|^{r-2} (\mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2 + \mu(r-2) |\nabla |u||^2).$$

For any given  $\epsilon_1 \in (0, 1)$  and  $\epsilon_0 \in (0, \frac{1}{4})$ , we define a nonnegative function which will be determined in **Step 2** as follows

$$\phi(\epsilon_0, \epsilon_1, r) = \begin{cases} \frac{\mu \epsilon_1 (r-1)}{3(-\frac{(4-\epsilon_0)\mu}{3} - \lambda + \frac{r^2(\lambda+\mu)}{4(r-1)})}, & \text{if } \frac{r^2(\mu+\lambda)}{4(r-1)} - \frac{(4-\epsilon_0)\mu}{3} - \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$

**Step 1:** we assume that

$$\int_{\Omega \cap \{|u|>0\}} |u|^r \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 dx > \phi(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 dx. \quad (3.8)$$

A direct calculation gives for  $|u| > 0$ :

$$|\nabla u|^2 = |u|^2 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u||^2, \quad (3.9)$$

which plays an important role in the proof. By (3.7) and the Cauchy's inequality, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho |u|^r dx + \int_{\Omega \cap \{|u|>0\}} H_r dx \\ &= -r(r-2)(\mu + \lambda) \int_{\Omega \cap \{|u|>0\}} \operatorname{div} u |u|^{\frac{r-2}{2}} |u|^{\frac{r-4}{2}} u \cdot \nabla |u| dx \\ & \quad + \int_{\Omega} r P \operatorname{div} (|u|^{r-2} u) dx - \int_{\Omega} r (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla (|u|^{r-2} u) dx \\ &\leq r(\mu + \lambda) \int_{\Omega \cap \{|u|>0\}} |u|^{r-2} |\operatorname{div} u|^2 dx + \frac{r(r-2)^2(\mu + \lambda)}{4} \int_{\Omega \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 dx \\ & \quad + \int_{\Omega} r P \operatorname{div} (|u|^{r-2} u) dx - \int_{\Omega} r (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla (|u|^{r-2} u) dx. \end{aligned} \quad (3.10)$$



Via Hölder's inequality, Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\begin{aligned}
J_1 &= \int_{\Omega} r P \operatorname{div} (|u|^{r-2} u) dx \\
&\leq Cr(r-1) \left( \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |u|^{r-2} P^2 dx \right)^{\frac{1}{2}} \\
&\leq Cr(r-1) |P|_{\frac{12r}{4r+4}} \left( \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (|u|^{\frac{r}{2}})^6 dx \right)^{\frac{2(r-2)}{12r}} \\
&\leq Cr(r-1) \left( \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (\nabla |u|^{\frac{r}{2}})^2 dx \right)^{\frac{(r-2)}{2r}} \\
&\leq \frac{1}{2} \mu r \epsilon_0 \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx + C(\mu, r, \epsilon_0), \\
J_2 &= - \int_{\Omega} r (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla (|u|^{r-2} u) dx \\
&\leq Cr(r-1) \left( \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |u|^{r-2} |H|^4 dx \right)^{\frac{1}{2}} \\
&\leq Cr(r-1) |H^2|_{\frac{12r}{4r+4}} \left( \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (|u|^{\frac{r}{2}})^6 dx \right)^{\frac{2(r-2)}{12r}} \\
&\leq Cr(r-1) \left( \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (\nabla |u|^{\frac{r}{2}})^2 dx \right)^{\frac{(r-2)}{2r}} \\
&\leq \frac{1}{2} \mu r \epsilon_0 \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx + C(\mu, r, \epsilon_0),
\end{aligned} \tag{3.11}$$

where  $\epsilon_0 \in (0, \frac{1}{4})$  is independent of  $r$ . Then combining (3.9)-(3.11), we quickly have

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega} \rho |u|^r dx + \int_{\Omega \cap \{|u|>0\}} \mu r (1 - \epsilon_0) |u|^{r-2} |\nabla |u||^2 dx \\
&\quad + \int_{\Omega \cap \{|u|>0\}} \mu r (1 - \epsilon_0) |u|^r \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 dx + \int_{\Omega \cap \{|u|>0\}} \mu r (r-2) |\nabla |u||^2 dx \\
&\leq \frac{r(r-2)^2(\mu + \lambda)}{4} \int_{\Omega \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 dx + C(\mu, r, \epsilon_0).
\end{aligned} \tag{3.12}$$

So according to (3.8) and (3.12), we obtain that

$$\frac{d}{dt} \int_{\Omega} \rho |u|^r dx + r f(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 dx \leq C(\mu, r, \epsilon_0), \tag{3.13}$$

where

$$f(\epsilon_0, \epsilon_1, r) = \mu(1 - \epsilon_0) \phi(\epsilon_0, \epsilon_1, r) + \mu(r - 1 - \epsilon_0) - \frac{(r-2)^2(\mu + \lambda)}{4}. \tag{3.14}$$

**Subcase 1:** if  $3 \in \left\{ r \left| \frac{r^2(\mu + \lambda)}{4(r-1)} - \frac{(4-\epsilon_0)\mu}{3} - \lambda > 0 \right. \right\}$ , i.e,  $(5 - 8\epsilon_0)\mu < 3\lambda$ , it is easy to get

$$[3, +\infty) \in \left\{ r \left| \frac{r^2(\mu + \lambda)}{4(r-1)} - \frac{(4-\epsilon_0)\mu}{3} - \lambda > 0 \right. \right\}.$$

Therefore, we have

$$\phi(\epsilon_0, \epsilon_1, r) = \frac{\mu\epsilon_1(r-1)}{3\left(-\frac{(4-\epsilon_0)\mu}{3} - \lambda + \frac{r^2(\lambda+\mu)}{4(r-1)}\right)} \quad (3.15)$$

for any  $r \in [3, \infty)$ . Substituting (3.15) into (3.14), for  $r \in [3, \infty)$ , we have

$$f(\epsilon_0, \epsilon_1, r) = \frac{\mu^2\epsilon_1(1-\epsilon_0)(r-1)}{3\left(-\frac{(4-\epsilon_0)\mu}{3} - \lambda + \frac{r^2(\lambda+\mu)}{4(r-1)}\right)} + \mu(r-1-\epsilon_0) - \frac{(r-2)^2(\mu+\lambda)}{4}. \quad (3.16)$$

For  $(\epsilon_1, r) = (1, 3)$ , we have

$$f(\epsilon_0, 1, 3) = \frac{16\mu^2(1-\epsilon_0)}{3\lambda - (5-8\epsilon_0)\mu} + \mu(2-\epsilon_0) - \frac{\mu+\lambda}{4} = -C_1(\lambda - a_1\mu)(\lambda - a_2\mu), \quad (3.17)$$

then according to  $\frac{(5-8\epsilon_0)\mu}{3} < \lambda$ , we have  $C_1 = \frac{3}{4(3\lambda - (5-8\epsilon_0)\mu)} > 0$  and

$$\begin{aligned} a_1 &= \frac{13 - 10\epsilon_0 + 2\sqrt{64 + \epsilon_0^2 - 56\epsilon_0}}{3}, \\ a_2 &= \frac{13 - 10\epsilon_0 - 2\sqrt{64 + \epsilon_0^2 - 56\epsilon_0}}{3} < 0. \end{aligned} \quad (3.18)$$

Then if we want to make sure that  $f(\epsilon_0, 1, 3) > 0$ , we have to assume that

$$\frac{(5-8\epsilon_0)\mu}{3} < \lambda < a_1\mu. \quad (3.19)$$

Due to  $a_1(0) = \frac{29}{3}$  and  $a'_1(\epsilon) < 0$  for  $\epsilon_0 \in (0, 1/4)$ , so there must exists a sufficiently small  $\epsilon_0 \in (0, 1/4)$  such that  $a_1(\epsilon_0) = \frac{29-\alpha_\lambda}{3}$ .

Since  $f(\epsilon_0, \epsilon_1, r)$  is continuous w.r.t.  $(\epsilon_1, r)$  over  $[0, 1] \times [3, +\infty)$ , there exists  $\epsilon_1 \in (0, 1)$  and  $r \in (3, \frac{7}{2})$ , such that  $f(\epsilon_0, \epsilon_1, r) \geq 0$ , which, together with (3.13)-(3.14), implies that

$$\frac{d}{dt} \int_{\Omega} \rho |u|^r dx \leq C, \quad \text{for } r \in (3, 7/2). \quad (3.20)$$

**Subcase 2:** if  $3 \notin \left\{r \left| \frac{r^2(\mu+\lambda)}{4(r-1)} - \frac{(4-\epsilon_0)\mu}{3} - \lambda > 0 \right.\right\}$ , i.e.,  $(5-8\epsilon_0)\mu \geq 3\lambda$ . In this case, for  $r \in (3, \frac{7}{2})$ , it is easy to get

$$\begin{aligned} & r \left[ \mu(1-\epsilon_0)\phi(\epsilon_0, \epsilon_1, r) + \mu(r-1-\epsilon_0) - \frac{(r-2)^2(\mu+\lambda)}{4} \right] \\ & > 3 \left( \frac{7}{4}\mu - \frac{9(\mu+\lambda)}{16} \right) = 3 \left( \frac{19\mu}{16} - \frac{9\lambda}{16} \right) \geq 3 \left( \frac{19\mu}{16} - \frac{3(5-8\epsilon_0)\mu}{16} \right) > \frac{1}{4}\mu, \end{aligned} \quad (3.21)$$

which, together with (3.13)-(3.14), implies that

$$\frac{d}{dt} \int_{\Omega} \rho |u|^r dx + \frac{1}{4}\mu \int_{\Omega \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 dx \leq C, \quad \text{for } r \in (3, 7/2). \quad (3.22)$$

**Step 2 :** we assume that

$$\int_{\Omega \cap \{|u|>0\}} |u|^r \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 dx \leq \phi(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 dx. \quad (3.23)$$

A direct calculation gives for  $|u| > 0$ ,

$$\operatorname{div} u = |u| \operatorname{div} \left( \frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|}. \quad (3.24)$$

Then combining (3.24) and (3.10)-(3.11), we quickly have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho |u|^r dx + \int_{\Omega \cap \{|u| > 0\}} \mu r (1 - \epsilon_0) |u|^{r-2} |\nabla u|^2 dx \\ & + \int_{\Omega \cap \{|u| > 0\}} r (\lambda + \mu) |u|^{r-2} |\operatorname{div} u|^2 dx + \int_{\Omega \cap \{|u| > 0\}} \mu r (r - 2) |u|^{r-2} |\nabla |u||^2 dx \\ & = -r(r-2)(\mu + \lambda) \int_{\Omega \cap \{|u| > 0\}} \left( |u|^{r-2} u \cdot \nabla |u| \operatorname{div} \left( \frac{u}{|u|} \right) + |u|^{r-4} |u \cdot \nabla |u||^2 \right) dx. \end{aligned} \quad (3.25)$$

This gives

$$\frac{d}{dt} \int_{\Omega} \rho |u|^r dx + \int_{\Omega \cap \{|u| > 0\}} r |u|^{r-4} G dx \leq C(\mu, r, \epsilon_0), \quad (3.26)$$

where

$$\begin{aligned} G = & \mu(1 - \epsilon_0) |u|^2 |\nabla u|^2 + (\mu + \lambda) |u|^2 |\operatorname{div} u|^2 + \mu(r-2) |u|^2 |\nabla |u||^2 \\ & + (r-2)(\mu + \lambda) |u|^2 u \cdot \nabla |u| \operatorname{div} \left( \frac{u}{|u|} \right) + (r-2)(\mu + \lambda) |u \cdot \nabla |u||^2. \end{aligned} \quad (3.27)$$

Now we consider how to make sure that  $G \geq 0$ .

$$\begin{aligned} G = & \mu(1 - \epsilon_0) |u|^2 \left( |u|^2 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u||^2 \right) + (\mu + \lambda) |u|^2 \left( |u| \operatorname{div} \left( \frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|} \right)^2 \\ & + \mu(r-2) |u|^2 |\nabla |u||^2 + (r-2)(\mu + \lambda) |u|^2 u \cdot \nabla |u| \operatorname{div} \left( \frac{u}{|u|} \right) \\ & + (r-2)(\mu + \lambda) |u \cdot \nabla |u||^2 \\ = & \mu(1 - \epsilon_0) |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + \mu(r-1 - \epsilon_0) |u|^2 |\nabla |u||^2 + (r-1)(\mu + \lambda) |u \cdot \nabla |u||^2 \\ & + r(\mu + \lambda) |u|^2 u \cdot \nabla |u| \operatorname{div} \left( \frac{u}{|u|} \right) + (\mu + \lambda) |u|^4 \left( \operatorname{div} \left( \frac{u}{|u|} \right) \right)^2 \\ = & \mu(1 - \epsilon_0) |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + \mu(r-1 - \epsilon_0) |u|^2 |\nabla |u||^2 \\ & + (r-1)(\mu + \lambda) \left( u \cdot \nabla |u| + \frac{r}{2(r-1)} |u|^2 \left( \operatorname{div} \frac{u}{|u|} \right) \right)^2 \\ & + (\mu + \lambda) |u|^4 \left( \operatorname{div} \frac{u}{|u|} \right)^2 - \frac{r^2(\mu + \lambda)}{4(r-1)} |u|^4 \left( \operatorname{div} \left( \frac{u}{|u|} \right) \right)^2, \end{aligned} \quad (3.28)$$

which, combining with the fact

$$\left| \operatorname{div} \left( \frac{u}{|u|} \right) \right|^2 \leq 3 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2,$$

implies that

$$\begin{aligned}
G &\geq \mu(1 - \epsilon_0)|u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + \mu(r - 1 - \epsilon_0)|u|^2 |\nabla |u||^2 \\
&\quad + \left( \mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right) |u|^4 \left( \operatorname{div} \left( \frac{u}{|u|} \right) \right)^2 \\
&\geq \frac{\mu(1 - \epsilon_0)}{3} |u|^4 \left| \operatorname{div} \left( \frac{u}{|u|} \right) \right|^2 + \mu(r - 1 - \epsilon_0)|u|^2 |\nabla |u||^2 \\
&\quad + \left( \mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right) |u|^4 \left( \operatorname{div} \left( \frac{u}{|u|} \right) \right)^2 \\
&\geq \mu(r - 1 - \epsilon_0)|u|^2 |\nabla |u||^2 + \left( \frac{(4 - \epsilon_0)\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right) |u|^4 \left( \operatorname{div} \left( \frac{u}{|u|} \right) \right)^2.
\end{aligned} \tag{3.29}$$

Thus

$$\begin{aligned}
&\int_{\Omega \cap \{|u| > 0\}} r|u|^{r-4} G dx \\
&\geq r \left( \frac{(4 - \epsilon_0)\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right) \int_{\Omega \cap \{|u| > 0\}} |u|^r \left( \operatorname{div} \left( \frac{u}{|u|} \right) \right)^2 dx \\
&\quad + \mu r(r - 1 - \epsilon_0) \int_{\Omega \cap \{|u| > 0\}} |u|^{r-2} |\nabla |u||^2 dx \\
&\geq 3r \left( \frac{(4 - \epsilon_0)\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right) \phi(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap \{|u| > 0\}} |u|^{r-2} |\nabla |u||^2 dx \\
&\quad + \mu r(r - 1 - \epsilon_0) \int_{\Omega \cap \{|u| > 0\}} |u|^{r-2} |\nabla |u||^2 dx \\
&\geq g(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap \{|u| > 0\}} |u|^{r-2} |\nabla |u||^2 dx,
\end{aligned} \tag{3.30}$$

where

$$g(\epsilon_0, \epsilon_1, r) = \left[ 3r \left( \frac{(4 - \epsilon_0)\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right) \phi(\epsilon_0, \epsilon_1, r) + \mu r(r - 1 - \epsilon_0) \right]. \tag{3.31}$$

Here we need that  $\epsilon_0$  is sufficiently small such that  $\epsilon_0 < (r - 1)(1 - \epsilon_1)$ . Then combining (3.26) and (3.30)-(3.31), we quickly have

$$\frac{d}{dt} \int_{\Omega} \rho |u|^r dx \leq C, \quad \text{for some } r \in (3, 7/2). \tag{3.32}$$

So combining (3.20)-(3.22) and (3.32) for **Step:1** and **Step:2**, we conclude that if  $3\lambda < (29 - \alpha_\lambda)\mu$ , there exists some constants  $C > 0$  such that

$$\frac{d}{dt} \int_{\Omega} \rho |u|^r dx \leq C, \quad \text{for some } r \in (3, 7/2). \tag{3.33}$$

□

Now for each  $t \in [0, T)$ , we denote  $v(t, x) = (-L)^{-1} \operatorname{div} A$  and

$$A = P I_3 - \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right),$$

that is,  $v$  is the solution of

$$\begin{cases} \mu \Delta v + (\lambda + \mu) \nabla \operatorname{div} v = \operatorname{div} A & \text{in } \Omega, \\ v(t, x) = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.34)$$

From Lemma 2.2, for any  $l \in (1, +\infty)$ , there exists a constant  $C$  independent of  $t$  such that

$$\begin{cases} |\nabla v(t)|_l \leq C(|\rho(t)|_l + |H(t)|_l), \\ |\nabla^2 v(t)|_l \leq C(|\nabla \rho(t)|_l + |\nabla H(t)|_l). \end{cases} \quad (3.35)$$

Now let us introduce an important quantity:

$$w = u - v.$$

It will be observed that this quantity  $w$  possesses more regularity information than  $u$  under the assumption that  $(H, \rho)$  is upper bounded. First, we have

**Lemma 3.3.**

$$|\nabla w(t)|_2^2 + \int_0^T (|\nabla^2 w|_2^2 + |\sqrt{\rho} w_t|_2^2) dt \leq C, \quad \text{for } 0 \leq t \leq T,$$

where  $C$  only depends on  $C_0, \mu, \lambda, A, \gamma, \Omega$  and  $T$  (any  $T \in (0, \overline{T}]$ ).

*Proof.* First, from the momentum equations (1.5)<sub>4</sub>, we find that  $w$  satisfies

$$\begin{cases} \rho w_t - \mu \Delta w - (\lambda + \mu) \nabla \operatorname{div} w = \rho F, \\ w(t, x) = 0 & \text{on } [0, T) \times \partial\Omega, \quad w(0, x) = w_0(x), \quad \text{in } \Omega, \end{cases} \quad (3.36)$$

with  $w_0(x) = u_0(x) + v_0(x)$  and

$$\begin{aligned} F &= -u \cdot \nabla u + L^{-1} \operatorname{div} A_t \\ &= -u \cdot \nabla u - L^{-1} \nabla \operatorname{div}(Pu) - (\gamma - 1) L^{-1} \nabla (P \operatorname{div} u) \\ &\quad - L^{-1} \operatorname{div}(H_t \otimes H + H \otimes H_t) + L^{-1} \nabla (H \cdot H_t) = \sum_{i=1}^5 L_i. \end{aligned}$$

Multiplying the equations in (3.36) by  $w_t$  and integrating the resulting equation over  $\Omega$ , from Hölder's inequality, we have

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} (\mu |\nabla w|^2 + (\lambda + \mu) |\operatorname{div} w|^2) dx + \int_{\Omega} \rho |w_t|^2 dx \\ &= \int_{\Omega} \rho F \cdot w_t dx \leq C |\sqrt{\rho} F|_2^2 + \frac{1}{2} \int_{\Omega} \rho |w_t|^2 dx, \end{aligned} \quad (3.37)$$

which means that

$$\frac{d}{dt} \int_{\Omega} (\mu |\nabla w|^2 + (\lambda + \mu) |\operatorname{div} w|^2) dx + \int_{\Omega} \rho |w_t|^2 dx \leq C \sum_{i=1}^5 |\sqrt{\rho} L_i|_2^2. \quad (3.38)$$

Next we need to consider the terms  $|\sqrt{\rho}L_i|_2$  for  $i = 1, 2, \dots, 5$ . From Lemma 3.2 and (3.35), it follows that

$$\begin{aligned} |\sqrt{\rho}L_1|_2 &= |-\sqrt{\rho}u \cdot \nabla u|_2 \leq C|\sqrt{\rho}u|_r |\nabla u|_{\frac{2r}{r-2}} \\ &\leq C\left(|\nabla w|_{\frac{2r}{r-2}} + |\nabla v|_{\frac{2r}{r-2}}\right) \leq C(\epsilon)|\nabla w|_2 + \epsilon|w|_{D^2} + C, \end{aligned} \quad (3.39)$$

where we have used the interpolation inequality

$$|f|_p \leq C(\epsilon)|f|_2 + \epsilon|\nabla f|_2, \quad 2 \leq p < 6.$$

According to Lemmas 3.1-3.2, we obtain

$$\begin{aligned} |\sqrt{\rho}L_2|_2 &= |-\sqrt{\rho}L^{-1}\nabla\operatorname{div}(Pu)|_2 \leq C|Pu|_2 \leq C|\sqrt{\rho}u|_2 \leq C, \\ |\sqrt{\rho}L_3|_2 &= |-(\gamma-1)\sqrt{\rho}L^{-1}\nabla(P\operatorname{div}u)|_2 \\ &\leq C|\sqrt{\rho}|_3 |L^{-1}\nabla(P\operatorname{div}u)|_6 \\ &\leq C|\nabla L^{-1}\nabla(P\operatorname{div}u)|_2 \leq C|P\operatorname{div}u|_2 \leq C|\nabla u|_2. \end{aligned} \quad (3.40)$$

Now we consider the term  $\mathbb{B} = (b^{(i,j)})_{(3 \times 3)} = H_t \otimes H + H \otimes H_t$ . Due to Lemma 2.5,

$$H_t = H \cdot \nabla u - u \cdot \nabla H - H\operatorname{div}u, \quad (3.41)$$

we get

$$\begin{aligned} b^{(i,j)} &= H^j (H^k \partial_k u^i - u^k \partial_k H^i - H^i \partial_k u^k) \\ &\quad + H^i (H^k \partial_k u^j - u^k \partial_k H^j - H^j \partial_k u^k) \\ &= H^i H^k \partial_k u^j + H^j H^k \partial_k u^i - H^i H^j \partial_k u^k - \partial_k (H^i H^j u^k), \end{aligned} \quad (3.42)$$

which means that

$$\mathbb{B} = (H^i H^k \partial_k u^j + H^j H^k \partial_k u^i - H^i H^j \partial_k u^k)_{(3 \times 3)} - \operatorname{div}((H \otimes H) \otimes u) = \mathbb{B}_1 + \mathbb{B}_2. \quad (3.43)$$

Then we have

$$\begin{aligned} |\sqrt{\rho}L_4|_2 &= |\sqrt{\rho}L^{-1}\operatorname{div}(H_t \otimes H + H \otimes H_t)|_2 \\ &= |\sqrt{\rho}L^{-1}\operatorname{div}\mathbb{B}_1|_2 + |\sqrt{\rho}L^{-1}\operatorname{div}\mathbb{B}_2|_2 \\ &\leq C|\sqrt{\rho}|_3 |L^{-1}\operatorname{div}\mathbb{B}_1|_6 + |\sqrt{\rho}L^{-1}\operatorname{div}\operatorname{div}((H \otimes H) \otimes u)|_2 \\ &\leq C|\nabla L^{-1}\operatorname{div}\mathbb{B}_1|_2 + C|\nabla u|_2 \leq C|\nabla u|_2. \end{aligned} \quad (3.44)$$

Similarly, we consider the term  $\mathbb{C} = H \cdot H_t$ . Due to (3.41), we obtain

$$\begin{aligned} \mathbb{C} &= H \cdot (H \cdot \nabla u - u \cdot \nabla H - H\operatorname{div}u) \\ &= \left(H \cdot \nabla u \cdot H - \frac{1}{2}|H|^2\operatorname{div}u\right) - \frac{1}{2}\operatorname{div}(u|H|^2) = \mathbb{C}_1 + \mathbb{C}_2, \end{aligned} \quad (3.45)$$

which, together with the Poincaré inequality, implies that

$$\begin{aligned} |\sqrt{\rho}L_5|_2 &= |\sqrt{\rho}L^{-1}\nabla(H \cdot H_t)|_2 \\ &= |\sqrt{\rho}L^{-1}\nabla\mathbb{C}_1|_2 + |\sqrt{\rho}L^{-1}\nabla\mathbb{C}_2|_2 \\ &\leq C|\sqrt{\rho}|_3 |L^{-1}\nabla\mathbb{C}_1|_6 + |\sqrt{\rho}L^{-1}\nabla\operatorname{div}(u|H|^2)|_2 \\ &\leq C|\nabla L^{-1}\nabla\mathbb{C}_1|_2 + C|\nabla u|_2 \leq C|\nabla u|_2. \end{aligned} \quad (3.46)$$

Combining (3.39)-(3.46), we have

$$|\sqrt{\rho}F|_2^2 \leq \epsilon |\nabla^2 w|_2^2 + C(\epsilon)(1 + |\nabla w|_2^2 + |\nabla u|_2^2). \quad (3.47)$$

Then from Lemma 2.2 and (3.36), we have

$$|\nabla^2 w|_2^2 \leq C(|\rho w_t|_2^2 + |\rho F|_2^2) \leq C(|\sqrt{\rho}w_t|_2^2 + |\sqrt{\rho}F|_2^2), \quad (3.48)$$

which implies, by taking  $\epsilon = \frac{1}{3C}$  in (3.47), that

$$|\sqrt{\rho}F|_2^2 \leq \frac{1}{2}|\sqrt{\rho}w_t|_2^2 + C(\epsilon)(1 + |\nabla w|_2^2 + |\nabla u|_2^2). \quad (3.49)$$

Substituting (3.48) into (3.38), from Gronwall's inequality, the desired conclusions can be obtained.  $\square$

Finally, according to the estimates obtained in (3.35) and Lemmas 3.2-3.3, we get

**Lemma 3.4.**

$$|\nabla u(t)|_2^2 + \int_0^T |\nabla u|_q^2 dt \leq C, \quad \text{for } 0 \leq t \leq T,$$

where  $C$  only depends on  $C_0, \mu, \lambda, A, \gamma, \Omega$  and  $T$  (any  $T \in (0, \overline{T})$ ).

Next, we will give high order regularity estimates for  $w$ . This is possible if the initial data  $(H_0, \rho_0, u_0, P_0)$  satisfies the compatibility condition (1.8). First for a function or vector field (or even a  $3 \times 3$  matrix)  $f(t, x)$ , the material derivative  $\dot{f}$  is defined by:

$$\dot{f} = f_t + u \cdot \nabla f = f_t + \operatorname{div}(fu) - f \operatorname{div} u.$$

**Lemma 3.5 (Lower order estimate of the velocity  $u$ ).**

$$|w(t)|_{D^2}^2 + |\sqrt{\rho}\dot{u}(t)|_2^2 + \int_0^T |\dot{u}|_{D^1}^2 dt \leq C, \quad \text{for } 0 \leq t \leq T,$$

where  $C$  only depends on  $C_0, \mu, \lambda, A, \gamma, \Omega$  and  $T$  (any  $T \in (0, \overline{T})$ ).

*Proof.* We will follow an idea due to Hoff [17]. Applying  $\dot{u}[\partial/\partial t + \operatorname{div}(u \cdot)]$  to (1.5)<sub>4</sub> and integrating by parts give

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\dot{u}|^2 dx \\ &= - \int_{\Omega} \left( \dot{u} \cdot (\nabla P_t + \operatorname{div}(\nabla P \otimes u)) + \dot{u} \cdot (\Delta u_t + \operatorname{div}(\Delta u \otimes u)) \right) dx \\ & \quad + (\lambda + \mu) \int_{\Omega} \dot{u} \cdot (\nabla \operatorname{div} u_t + \operatorname{div}(\nabla \operatorname{div} u \otimes u)) dx \\ & \quad + \int_{\Omega} \dot{u} \cdot \left( \operatorname{div} \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right)_t + \operatorname{div} \left( \operatorname{div} \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right) \otimes u \right) \right) dx \\ & \equiv : \sum_{i=6}^8 L_i + \Lambda. \end{aligned} \quad (3.50)$$

According to Lemmas 3.1-3.4, Hölder's inequality, Gagliardo-Nirenberg inequality and Young's inequality, we deduce that

$$\begin{aligned}
L_6 &= - \int_{\Omega} (\dot{u} \cdot (\nabla P_t + \operatorname{div}(\nabla P \otimes u))) dx \\
&= \int_{\Omega} (\partial_j \dot{u}^j P_t + \partial_k \dot{u}^j \partial_j P u^k) dx \\
&= \int_{\Omega} (-\partial_j \dot{u}^j u^k \partial_k P - \gamma P \operatorname{div} u \partial_j \dot{u}^j + \partial_k \dot{u}^j \partial_j P u^k) dx \\
&= \int_{\Omega} (-\gamma P \operatorname{div} u \partial_j \dot{u}^j + P \partial_k (\partial_j \dot{u}^j u^k) - P \partial_j (\partial_k \dot{u}^j u^k)) dx \\
&\leq C |\nabla \dot{u}|_2 |\nabla u|_2 \leq \epsilon |\nabla \dot{u}|_2^2 + C(\epsilon) |\nabla u|_2^2, \\
L_7 &= \int_{\Omega} \mu (\dot{u} \cdot (\Delta u_t + \operatorname{div}(\Delta u \otimes u))) dx \\
&= - \int_{\Omega} \mu (\partial_i \dot{u}^j \partial_i u_t^j + \Delta u^j u \cdot \nabla \dot{u}^j) dx \\
&= - \int_{\Omega} \mu (|\nabla \dot{u}|^2 - \partial_i \dot{u}^j u^k \partial_k \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j + \Delta u^j u \cdot \nabla \dot{u}^j) dx \\
&= - \int_{\Omega} \mu (|\nabla \dot{u}|^2 - \partial_i \dot{u}^j \partial_k u^k \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j - \partial_i u^j \partial_i u^k \partial_k \dot{u}^j) dx \\
&\leq -\frac{\mu}{2} |\nabla \dot{u}|_2^2 + C |\nabla u|_4^4,
\end{aligned} \tag{3.51}$$

where  $\epsilon > 0$  is a sufficiently small constant. Similarly, we have

$$L_8 = (\lambda + \mu) \int_{\Omega} (\dot{u} \cdot (\nabla \operatorname{div} u_t + \operatorname{div}(\nabla \operatorname{div} u \otimes u))) dx \leq -\frac{\mu + \lambda}{2} |\nabla \dot{u}|_2^2 + C |\nabla u|_4^4. \tag{3.52}$$

Next we begin to consider the magnetic term  $\Lambda$

$$\Lambda = \int_{\Omega} \dot{u} \cdot \left( \operatorname{div} \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right)_t + \operatorname{div} \left( \operatorname{div} \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right) \otimes u \right) \right) dx = \sum_{j=1}^4 \Lambda_j.$$

Via the magnetic equations (1.5)<sub>1</sub> and integrating by parts, we obtain that

$$\begin{aligned}
\Lambda_1 &= \int_{\Omega} \dot{u} \cdot \operatorname{div} (H \otimes H)_t dx = \int_{\Omega} (H \otimes H)_t : \nabla \dot{u} dx \\
&= \int_{\Omega} (H \otimes H_t + H_t \otimes H) : \nabla \dot{u} dx \\
&= \int_{\Omega} H \otimes (H \cdot \nabla u - u \cdot \nabla H - H \operatorname{div} u) : \nabla \dot{u} dx \\
&\quad + \int_{\Omega} (H \cdot \nabla u - u \cdot \nabla H - H \operatorname{div} u) \otimes H : \nabla \dot{u} dx \\
&= \int_{\Omega} H \otimes (H \cdot \nabla u - H \operatorname{div} u) : \nabla \dot{u} dx + \int_{\Omega} (H \cdot \nabla u - H \operatorname{div} u) \otimes H : \nabla \dot{u} dx \\
&\quad + \int_{\Omega} \left( -H \otimes (u \cdot \nabla H) - (u \cdot \nabla H) \otimes H \right) : \nabla \dot{u} dx = \Lambda_{11} + \Lambda_{12} + \Lambda_{13},
\end{aligned} \tag{3.53}$$



$$\begin{aligned}
\Lambda_2 &= \int_{\Omega} \dot{u} \cdot \operatorname{div} \left( -\frac{1}{2} |H|^2 I_3 \right)_t dx = \int_{\Omega} \left( -\frac{1}{2} |H|^2 I_3 \right)_t : \nabla \dot{u} dx \\
&= \int_{\Omega} -(H \cdot H_t I_3) : \nabla \dot{u} dx \\
&= \int_{\Omega} -(H \cdot (H \cdot \nabla u - u \cdot \nabla H - H \operatorname{div} u) I_3) : \nabla \dot{u} dx \\
&= \int_{\Omega} -(H \cdot (H \cdot \nabla u - H \operatorname{div} u) I_3) : \nabla \dot{u} dx + \int_{\Omega} (H \cdot (u \cdot \nabla H) I_3) : \nabla \dot{u} dx \\
&= \Lambda_{21} + \Lambda_{22}, \\
\Lambda_3 &= \int_{\Omega} \dot{u} \cdot \operatorname{div} (\operatorname{div} (H \otimes H) \otimes u) dx = \int_{\Omega} \operatorname{div} (H \otimes H) \otimes u : \nabla \dot{u} dx \\
&= \int_{\Omega} (H \cdot \nabla H) \otimes u : \nabla \dot{u} dx = \int_{\Omega} H^k \partial_k H^i u^j \partial_j \dot{u}^i dx \\
&= - \int_{\Omega} H^k H^i \partial_k u^j \partial_j \dot{u}^i dx - \int_{\Omega} H^k H^i u^j \partial_{kj} \dot{u}^i dx = \Lambda_{31} + \Lambda_{32}, \\
\Lambda_4 &= \int_{\Omega} \dot{u} \cdot \operatorname{div} \left( \operatorname{div} \left( -\frac{1}{2} |H|^2 I_3 \right) \otimes u \right) dx \\
&= \int_{\Omega} \operatorname{div} \left( -\frac{1}{2} |H|^2 I_3 \right) \otimes u : \nabla \dot{u} dx = \int_{\Omega} -H^k \partial_i H^k u^j \partial_j \dot{u}^i dx \\
&= \frac{1}{2} \int_{\Omega} |H^k|^2 \partial_i u^j \partial_j \dot{u}^i dx + \frac{1}{2} \int_{\Omega} |H^k|^2 u^j \partial_{ij} \dot{u}^i dx = \Lambda_{41} + \Lambda_{42},
\end{aligned} \tag{3.54}$$

where we have used the fact that  $\operatorname{div} H = 0$ . Now we observe that

$$\begin{aligned}
\Lambda_{13} + \Lambda_{32} &= \int_{\Omega} \left( -H \otimes (u \cdot \nabla H) - (u \cdot \nabla H) \otimes H \right) : \nabla \dot{u} dx - \int_{\Omega} H^k H^i u^j \partial_{kj} \dot{u}^i dx \\
&= \int_{\Omega} \left( -H^i u^k \partial_k H^j \partial_j \dot{u}^i - u^k \partial_k H^i H^j \partial_j \dot{u}^i - H^k H^i u^j \partial_{kj} \dot{u}^i \right) dx \\
&= \int_{\Omega} \left( \partial_k u^k H^i H^j \partial_j \dot{u}^i + u^k H^j \partial_k H^i \partial_j \dot{u}^i + H^j H^i u^k \partial_{kj} \dot{u}^i \right) dx \\
&\quad + \int_{\Omega} \left( -u^k H^j \partial_k H^i \partial_j \dot{u}^i - H^k H^i u^j \partial_{kj} \dot{u}^i \right) dx \\
&= \int_{\Omega} \partial_k u^k H^i H^j \partial_j \dot{u}^i dx, \\
\Lambda_{22} + \Lambda_{42} &= \int_{\Omega} (H \cdot (u \cdot \nabla H) I_3) : \nabla \dot{u} dx + \frac{1}{2} \int_{\Omega} |H^k|^2 u^j \partial_{ij} \dot{u}^i dx \\
&= \int_{\Omega} \left( H^k u^l \partial_l H^k \operatorname{div} \dot{u} + \frac{1}{2} |H^k|^2 u^j \partial_{ij} \dot{u}^i \right) dx \\
&= \int_{\Omega} \left( -\frac{1}{2} u^j |H^k|^2 \partial_{ij} \dot{u}^i - \frac{1}{2} \partial_j u^j |H^k|^2 \partial_i \dot{u}^i + \frac{1}{2} |H^k|^2 u^j \partial_{ij} \dot{u}^i \right) dx \\
&= -\frac{1}{2} \int_{\Omega} \partial_j u^j |H^k|^2 \partial_i \dot{u}^i dx,
\end{aligned} \tag{3.55}$$

which, together with (3.53)-(3.54), implies that

$$\Lambda \leq C|H|_\infty^2 |\nabla u|_2 |\nabla \dot{u}|_2 \leq \epsilon |\nabla \dot{u}|_2^2 + C(\epsilon) |\nabla u|_2^2. \quad (3.56)$$

Due to the definition of  $w$ , we know that  $w$  satisfies

$$\mu \Delta w + (\lambda + \mu) \nabla \operatorname{div} w = \rho \dot{u} \quad \text{in } \Omega, \quad (3.57)$$

with the zero boundary condition. From Lemma 2.2, we have

$$|w|_{D^2} \leq C |\rho \dot{u}|_2 \leq C |\rho \dot{u}|_2, \quad (3.58)$$

which, together with (3.50)-(3.56) and letting  $\epsilon > 0$  be sufficiently small, implies that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\dot{u}|^2 dx + |\dot{u}|_{D^1}^2 &\leq C |\nabla u|_4^4 + C \\ &\leq C |\nabla u|_2 |\nabla u|_6^3 \leq C |\nabla u|_6^2 (|\nabla w|_6 + |\nabla v|_6) \\ &\leq C |\nabla u|_6^2 (1 + |\nabla^2 w|_2) \leq C |\nabla u|_6^2 (1 + |\sqrt{\rho} \dot{u}|_2). \end{aligned} \quad (3.59)$$

Then from Gronwall's inequality, we have

$$\int_\Omega \rho |\dot{u}|^2(t) dx + \int_0^t |\dot{u}|_{D^1}^2 \leq C, \quad \text{for } 0 \leq t \leq T. \quad (3.60)$$

□

According to Lemmas 3.2-3.5 and using the equations (3.57) again, we deduce

**Lemma 3.6.**

$$|\nabla w(t)|_{L^2([0,T];L^\infty(\Omega))} + |\nabla^2 w(t)|_{L^2([0,T];L^q(\Omega))} \leq C, \quad \text{for } 0 \leq t \leq T,$$

where  $C$  only depends on  $C_0, \mu, \lambda, A, \gamma, \Omega$  and  $T$  (any  $T \in (0, \bar{T}]$ ).

Finally, the following lemma gives bounds of  $|\nabla \rho|_q, |\nabla H|_q$  and  $|\nabla^2 u|_q$ .

**Lemma 3.7.**

$$\|(\rho, H, P)(t)\|_{W^{1,r}} + |(\rho_t, H_t, P_t)(t)|_r \leq C, \quad \text{for } 0 \leq t \leq T, \quad (3.61)$$

where  $r \in [2, q]$ ,  $C$  only depends on  $C_0, \mu, \lambda, A, \gamma, \Omega$  and  $T$  (any  $T \in (0, \bar{T}]$ ).

*Proof.* In the following estimates we will use (from (3.34)-(3.35) and (2.10)-(??))

$$\begin{aligned} |\nabla^2 v|_q &\leq C(|\nabla \rho|_q + |\nabla H|_q), \\ |\nabla v|_\infty &\leq C(1 + |\nabla v|_{BMO(\Omega)} \ln(e + |\nabla^2 v|_q)) \\ &\leq C(1 + (|\rho|_{L^2 \cap L^\infty} + |H|_{L^2 \cap L^\infty}) \ln(e + |\nabla \rho|_q + |\nabla H|_q)) \\ &\leq C(1 + \ln(e + |\nabla \rho|_q + |\nabla H|_q)). \end{aligned} \quad (3.62)$$

First, applying  $\nabla$  to (1.5)<sub>3</sub>, multiplying the resulting equations by  $q|\nabla \rho|^{q-2} \nabla \rho$ , we have

$$\begin{aligned} &(|\nabla \rho|^q)_t + \operatorname{div}(|\nabla \rho|^q u) + (q-1)|\nabla \rho|^q \operatorname{div} u \\ &= -q|\nabla \rho|^{q-2} (\nabla \rho)^\top D(u) (\nabla \rho) - q\rho |\nabla \rho|^{q-2} \nabla \rho \cdot \nabla \operatorname{div} u. \end{aligned} \quad (3.63)$$

Then integrating (3.63) over  $\Omega$ , we immediately obtain

$$\begin{aligned} \frac{d}{dt} |\nabla \rho|_q^q &\leq C |D(u)|_\infty |\nabla \rho|_q^q + C |\nabla^2 u|_q |\nabla \rho|_q^{q-1} \\ &\leq C (|\nabla w|_\infty + |\nabla v|_\infty) |\rho|_q^q + C (|\nabla^2 w|_q + |\nabla^2 v|_q) |\nabla \rho|_q^{q-1}. \end{aligned} \quad (3.64)$$

Second, applying  $\nabla$  to (1.5)<sub>1</sub>, multiplying the resulting equations by  $q \nabla H |\nabla H|^{q-2}$ , we have

$$(|\nabla H|^2)_t - qA : \nabla H |\nabla H|^{q-2} + qB : \nabla H |\nabla H|^{q-2} + qC : \nabla H |\nabla H|^{q-2} = 0, \quad (3.65)$$

where

$$\begin{aligned} A &= \nabla(H \cdot \nabla u) = (\partial_j H \cdot \nabla u^i)_{(ij)} + (H \cdot \nabla \partial_j u^i)_{(ij)}, \\ B &= \nabla(u \cdot \nabla H) = (\partial_j u \cdot \nabla H^i)_{(ij)} + (u \cdot \nabla \partial_j H^i)_{(ij)}, \\ C &= \nabla(H \operatorname{div} u) = \nabla H \operatorname{div} u + H \otimes \nabla \operatorname{div} u. \end{aligned} \quad (3.66)$$

Then integrating (3.65) over  $\Omega$ , due to

$$\begin{aligned} \int_\Omega A : \nabla H |\nabla H|^{q-2} dx &\leq C |\nabla u|_\infty |\nabla H|_q^q + C |H|_\infty |\nabla H|_q^{q-1} |u|_{D^{2,q}}, \\ \int_\Omega B : \nabla H |\nabla H|^{q-2} dx \\ &= \int_\Omega \sum_{i,j,k} \partial_j u^k \partial_k H^i \partial_j H^i |\nabla H|^{q-2} dx + \int_\Omega \sum_{i,j,k} u^k (\partial_k H^i \partial_j H^i) |\nabla H|^{q-2} dx \\ &= C |\nabla u|_\infty |\nabla H|_q^q + \frac{1}{2} \int_\Omega \sum_{k=1}^3 u^k \left( \partial_k |\nabla H|^2 |\nabla H|^{q-2} \right) dx \\ &= C |\nabla u|_\infty |\nabla H|_q^q + \frac{1}{q} \int_\Omega \sum_{k=1}^3 u^k \partial_k |\nabla H|^q dx \leq C |\nabla u|_\infty |\nabla H|_q^q, \\ \int_\Omega C : \nabla H |\nabla H|^{q-2} dx &\leq C |\nabla u|_\infty |\nabla H|_q^q + C |H|_\infty |\nabla H|_q^{q-1} |u|_{D^{2,q}}, \end{aligned} \quad (3.67)$$

we quickly obtain the following estimate:

$$\begin{aligned} \frac{d}{dt} |\nabla H|_q^q &\leq C (|\nabla u|_\infty + 1) |\nabla H|_q^q + C |u|_{D^{2,q}} |\nabla H|_q^{q-1} \\ &\leq C (|\nabla w|_\infty + |\nabla v|_\infty) |\nabla H|_q^q + C (|\nabla^2 w|_q + |\nabla^2 v|_q) |\nabla H|_q^{q-1}. \end{aligned} \quad (3.68)$$

Then from (3.62), (3.64), (3.68) and Gronwall's inequality, we immediately have

$$\begin{aligned} &\frac{d}{dt} (|\nabla \rho|_q^q + |\nabla H|_q^q) \\ &\leq C (1 + |\nabla w|_\infty + |\nabla v|_\infty) (|\nabla \rho|_q^q + |\nabla H|_q^q) + C |\nabla^2 w|_q (|\nabla \rho|_q^{q-1} + |\nabla H|_q^{q-1}) \\ &\leq C (1 + \|\nabla w\|_{W^{1,q}} + \ln(e + |\nabla \rho|_q + |\nabla H|_q)) (|\nabla \rho|_q^q + |\nabla H|_q^q) \\ &\quad + C |\nabla^2 w|_q (|\nabla \rho|_q^{q-1} + |\nabla H|_q^{q-1}). \end{aligned} \quad (3.69)$$

Via (3.69) and notations:

$$f = e + |\nabla \rho|_q + |\nabla H|_q, \quad g = 1 + \|\nabla w\|_{W^{1,q}},$$

we quickly have

$$f_t \leq Cgf + Cf \ln f + Cg,$$

which, together with Lemma 3.6 and Gronwall's inequality, implies that

$$\ln f(t) \leq C, \quad \text{for } 0 \leq t \leq T.$$

Then we have obtained the desired estimate for  $|\nabla \rho|_q + |\nabla H|_q$ . And the upper bound of  $|\nabla \rho|_r + |\nabla H|_r$  can be deduced via the Hölder's inequality.

Finally, the estimates for  $\rho_t$  and  $H_t$  can be obtained easily via the following relation:

$$\begin{cases} H_t = H \cdot \nabla u - u \cdot \nabla H - H \operatorname{div} u, \\ \rho_t = -u \cdot \nabla \rho - \rho \operatorname{div} u, \quad P_t = -u \cdot \nabla P - \gamma P \operatorname{div} u, \end{cases} \quad (3.70)$$

and the estimates obtained in Lemmas 3.1-3.7.  $\square$

According to the estimates obtained in Lemmas 3.1-3.7, we deduce that

**Lemma 3.8.**

$$|u(t)|_{D^2} + |\sqrt{\rho}u(t)|_2 + \int_0^T (|u_t|_{D^1}^2 + |u|_{D^{2,q}}^2) dt \leq C, \quad \text{for } 0 \leq t \leq T,$$

where  $C$  only depends on  $C_0, \mu, \lambda, A, \gamma, \Omega$  and  $T$  (any  $T \in (0, \bar{T}]$ ).

*Proof.* Via the momentum equations (1.5)<sub>4</sub>, (3.35) and Lemma 2.2, we have

$$|u|_{D^{2,l}} \leq (|w|_{D^{2,l}} + |v|_{D^{2,l}}) \leq C(|w|_{D^{2,l}} + |\nabla P|_l + |\nabla H|_l),$$

which, together with Lemma 3.7, implies that

$$|u(t)|_{D^2} + \int_0^T |u|_{D^{2,q}}^2 dt \leq C, \quad \text{for } 0 \leq t \leq T.$$

According to Lemmas 3.2 and 3.7, for  $r \in (3, 7/2)$ , we quickly have

$$|\sqrt{\rho}u_t|_2 \leq C(|\sqrt{\rho}\dot{u}|_2 + |\sqrt{\rho}u \cdot \nabla u|_2) \leq C(1 + |\rho^{\frac{1}{r}}u|_r |\nabla u|_{\frac{2r}{r-2}}) \leq C.$$

Similarly, we have

$$\int_0^T |u_t|_{D^1}^2 dt \leq C \int_0^T (|\dot{u}|_{D^1}^2 + |u \cdot \nabla u|_{D^1}^2) dt \leq C.$$

$\square$

In truth, in view of the estimates obtained in Lemmas 3.1-3.8, we quickly know that the functions  $(H, \rho, u, P)|_{t=\bar{T}} = \lim_{t \rightarrow \bar{T}} (H, \rho, u, P)$  satisfies the conditions imposed on the initial data (1.7) – (1.8). Therefore, we can take  $(H, \rho, u, P)|_{t=\bar{T}}$  as the initial data and apply the local existence Theorem 1.1 to extend our local strong solution beyond  $t \geq \bar{T}$ . This contradicts the assumption on  $\bar{T}$ .

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